4. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify your answer.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim_{x \to -1}\left(\frac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

(a) $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right) = 2x - \frac{x^2}{2} + \frac{1}{4}y$

$2: \begin{cases} 1: & \text{implicit differentiation} \\ 1: & \text{answer} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2}(8) = 0$

$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2}(-2)^2 + \frac{1}{4}(8) = -4 > 0$

f has a relative maximum at (-2,8) because $\left.\frac{dy}{dx}\right|_{(-2,8)} = 0$

and is decreasing, going from positive to negative, which means that f goes from increasing to decreasing

$3: \begin{cases} 1: & \left.\frac{dy}{dx}\right|_{(-2,8)} \\ 1: & \left.\frac{d^2y}{dx^2}\right|_{(-2,8)} \\ 1: & \text{answer with justification} \end{cases}$

(c) $\lim_{x \to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right) = \frac{0}{0}$

Using L'Hospital's Rule:

$\lim_{x \to -1}\left(\frac{g'(x)}{6(x+1)}\right) = \frac{0}{0}$

UsingL'Hospital's Rule:

$\lim_{x \to -1}\left(\frac{g''(x)}{6}\right) = \frac{g''(-1)}{6} = \frac{2(-1) - \frac{1}{2}(1) + \frac{1}{4}(2)}{6} = -\frac{1}{3}$

$2: \begin{cases} 1: & \text{L'Hospital's Rule} \\ 1: & \text{answer} \end{cases}$

(d) $h(0) = \frac{1}{2}$

$h\left(\frac{1}{2}\right) = h(0) + h'(0)\left(\frac{1}{2}\right) = 2 + \left(0 - \frac{1}{2}(2)\right)\left(\frac{1}{2}\right) = \frac{3}{2}$

$h(1) = h\left(\frac{1}{2}\right) + h'\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{2} + \left(\frac{1}{4} - \frac{1}{2}\left(\frac{3}{2}\right)\right)\left(\frac{1}{2}\right) = \frac{5}{4}$

$2: \begin{cases} 1: & \text{Euler's Method} \\ 1: & \text{answer} \end{cases}$