AP® CALCULUS BC

2017 SCORING GUIDELINES

Question 6

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

   (a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

   (b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

   (c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

   (d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

   $$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

(a) $f(0) = 0$ $f'(0) = 1$ $f''(0) = -1$ $f^{(3)}(0) = 2$ $f^{(4)}(0) = -6$ $P_4(x)\grave{} = 0 + 1x + \frac{(-1)x^2}{2!} + \frac{2x^3}{3!} + \frac{(-6)x^4}{4!}$ $= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \frac{(-1)^{n+1}x^n}{n}$	3: { 1: Derivatives 1: first four non zero terms 1: general term
(b) The series when x=1 is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$. This is the Alternating Harmonic series, $\frac{(-1)^{n+1}}{n}$, which converges by the Alternating Series error bound. The	2: { 1: Determine convergence 1: Correct explanation of why

Harmonic, $\frac{1}{n}$, diverges since n is raised to a power less than two. Thus, this series is conditionally convergent because the absolute value of the series does not converge, even though the series converges.

(c) $\int_0^x f(t)\,dt = \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{12} - \frac{t^5}{20}\Big|_0^x$

$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^n(x^{n+2})}{(n+2)(n+1)}$ where $n \geq 0$

2: { 1 : first four terms
1 : general term }

(d) $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < P_5\left(\frac{1}{2}\right) < \frac{1}{500}$

$P_5\left(\frac{1}{2}\right) = \left|-\frac{\left(\frac{1}{2}\right)^5}{20}\right| = \frac{1}{(32)(20)} = \frac{1}{640}$

2: { 1 : stating that $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < P_5\left(\frac{1}{2}\right)$
1 : finding $P_5\left(\frac{1}{2}\right)$ }