AP® CALCULUS BC
2018 SCORING GUIDELINES

The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

(a) $\ln\left(1+\frac{x}{3}\right) = \frac{x}{3} - \frac{1}{2}\left(\frac{x}{3}\right)^2 + \frac{1}{3}\left(\frac{x}{3}\right)^3 - \frac{1}{4}\left(\frac{x}{3}\right)^4 + \cdots + (-1)^{n+1}\left(\frac{1}{n}\right)\left(\frac{x}{3}\right)^n + \cdots$

$x\ln\left(1+\frac{x}{3}\right) = \frac{x^2}{3} - \frac{x^3}{2\cdot 3^2} + \frac{x^4}{3\cdot 3^3} - \frac{x^5}{4\cdot 3^4} + \cdots + (-1)^{n+1}\cdot\frac{x^{n+1}}{3^n\cdot n} + \cdots$

2 : { 1 : first four terms; 1 : general term }

(b) $\left|\frac{x^{n+2}}{3^{n+1}(n+1)}\right| = \left|\frac{n}{3(n+1)}\cdot x\right|$

$\lim_{n\to\infty}\frac{n}{3(n+1)}\cdot x = \frac{x}{3}$

$\frac{x}{3} < 1 \Rightarrow -3 < x < 3$

The series converges when $-3 < x \le 3$

When $x = -3$, the series is $3 + \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \cdots$

The series diverges.

When $x = 3$, the series is $3 - \frac{3}{2} + \frac{3}{3} - \frac{3}{4} + \cdots$

The series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-3 < x \le 3$.

5 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : identifies interior of interval of convergence; 1 : considers both endpoints; 1 : analysis and interval of convergence }

(c) $|P_4(2) - f(2)| < \frac{2^6}{5\cdot 3^5} = \frac{64}{1215}$

2 : { 1 : uses the fifth term as an error bound; 1 : error bound }

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