HIGH SCHOOL MATHEMATICS CONTESTS

Math League Press, P.O. Box 17, Tenafly, New Jersey 07670-0017

All official participants must take this contest at the same time.

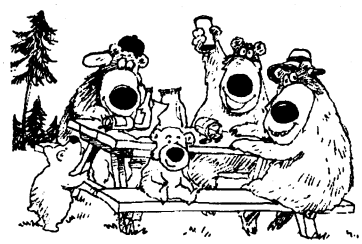
Contest Number 2 *Any calculator without a QWERTY keyboard is allowed.* Answers must be exact *or* have 4 (or more) significant digits, correctly rounded. **November 13, 2012**

Name ______________________ **Teacher** ___________ **Grade Level** _____ **Score** ____

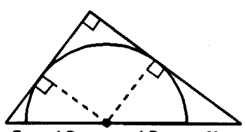
Time Limit: 30 minutes NEXT CONTEST: DEC. 11, 2012 *Answer Column*

2-1. What is the largest prime divisor of every 3-digit number with 3 identical non-zero digits? 2-1.

2-2. If 3 adult bears ate an average of 16 hot dogs each, and 2 bear cubs ate an average of 6 hot dogs each, then (for these 5 bears) what was the average number of hot dogs eaten per bear? 2-2.



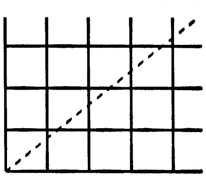
2-3. A semicircle is tangent to both legs of a right triangle and has its center on the hypotenuse. The hypotenuse is partitioned into 4 segments, with lengths 3, 12, 12, and x, as shown. What is the value of x? 2-3.



2-4. What are all 3 ordered triples of integers (a,b,c), with $0 < a \le b \le c$, for which $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$? 2-4.

2-5. A distribution consists of the integers from 1 through 100, inclusive, such that the frequency of each integer n is 2^{n-1}. What is the median of this distribution? 2-5.

2-6. If S is a 2012×2012 square split into unit squares, a diagonal of S will pass through the interior of 2012 unit squares. If R is a 2012×2015 rectangle split into unit squares, a diagonal of R will pass through the interior of how many unit squares? 2-6.



Eighteen books of past contests, *Grades 4, 5, & 6* (*Vols. 1, 2, 3, 4, 5, 6*), *Grades 7 & 8* (*Vols. 1, 2, 3, 4, 5, 6*), and *HS* (*Vols. 1, 2, 3, 4, 5, 6*), are available, for $12.95 each volume ($15.95 Canadian), from Math League Press, P.O. Box 17, Tenafly, NJ 07670-0017.

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Contest # 2 *Answers & Solutions* 11/13/12

Problem 2-1

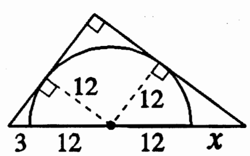
Every such 3-digit number *ddd* can be written as $d \times 111 = d \times 3 \times 37$. Since d is a digit, $1 \le d \le 9$, so the largest prime divisor is $\boxed{37}$.

Problem 2-2

The average of 5 numbers is their sum divided by 5. Since the sum of these 5 numbers is $(16+16+16) + (6+6) = 60$, their average is $60/5 = \boxed{12}$.

Problem 2-3

The two small right $\triangle$s are similar to each other (and the large right $\triangle$). A radius of the circle is 12. Thus the longer leg of the right $\triangle$ at the lower left is 12. Since its hypotenuse is 15, its dimensions are 9, 12, 15. The shorter leg of the right $\triangle$ at the lower right is 12, so its dimensions are 12, 16, 20. Since $12+x = 20$, $x = \boxed{8}$.



Problem 2-4

Clearly, $(a,b,c) = (3,3,3)$ is a solution. In any other solution, at least one fraction must exceed $\frac{1}{3}$, which means one fraction must equal $\frac{1}{2}$. Since $0 < a \le b \le c$, it follows that, in any other solution, $a = 2$. Now, solve $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ in positive integers. This is a simpler version of the original equation. This time, an obvious solution is $(b,c) = (4,4)$. In any other solution, one fraction must exceed $\frac{1}{4}$. That means that one fraction must equal $\frac{1}{3}$. Thus, $\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Finally, the positive integer solutions are the ordered triples $\boxed{(3,3,3),\ (2,4,4),\ (2,3,6)}$.

Problem 2-5

The integers range from 1 through 100. How many 1's are there? There's $2^{1-1} = 1$ of them. How many 2's? There are $2^1 = 2$ of those. Similarly, there are 2^2 3's, 2^3 4's, . . . , 2^{98} 99's. The total number of all these integers is $2^0+2^1+2^2+2^3+ \ldots +2^{98} = 2^{99}-1$. The number of 100's is 2^{99}, so we can pair one 100 with every other integer—and we'll still have one 100 left over. So, if the numbers are ordered from least to greatest, the middle number will be the extra $\boxed{100}$.

Problem 2-6

Place the rectangle on the coordinate axes with vertices (0,0), (2015,0), (2015,2012), and (0,2012). The diagonal is $y = \frac{2012}{2015}x$, with $0 \le x \le 2015$. The key observation is that the diagonal enters a new square each time the diagonal crosses a vertical line of the form $x = a$, with $a = 1, 2, 3, \ldots, 2014$, or a horizontal line of the form $y = b$, with $b = 1, 2, 3, \ldots, 2011$. (Since the greatest common divisor of 2012 and 2015 is 1, the diagonal never passes through any point with integral coordinates—where the grid lines cross—that is interior to the rectangle.) Start with the unit square that has one vertex at (0,0), then go through another 2014 squares horizontally and 2011 squares vertically. The total number of such squares is $1+2014+2011 = \boxed{4026}$.

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