Tuesday, FEBRUARY 12, 2008

59th Annual American Mathematics Contest 12

AMC 12
CONTEST A

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 26th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 18, 2008 or Wednesday, April 2, 2008. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. A bakery owner turns on his doughnut machine at 8:30 AM. At 11:10 AM the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

   (A) 1:50 PM **(B)** 3:00 PM **(C)** 3:30 PM **(D)** 4:30 PM **(E)** 5:50 PM

2. What is the reciprocal of $\frac{1}{2} + \frac{2}{3}$?

   (A) $\frac{6}{7}$ **(B)** $\frac{7}{6}$ **(C)** $\frac{5}{3}$ **(D)** 3 **(E)** $\frac{7}{2}$

3. Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

   (A) 2 **(B)** $\frac{5}{2}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** 4

4. Which of the following is equal to the product

$$\frac{8}{4} \cdot \frac{12}{8} \cdot \frac{16}{12} \cdot \cdots \cdot \frac{4n+4}{4n} \cdot \cdots \cdot \frac{2008}{2004}?$$

   (A) 251 **(B)** 502 **(C)** 1004 **(D)** 2008 **(E)** 4016

5. Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

   is an integer. Which of the following statements must be true about x?

   (A) It is negative. **(B)** It is even, but not necessarily a multiple of 3.

   (C) It is a multiple of 3, but not necessarily even.

   (D) It is a multiple of 6, but not necessarily a multiple of 12.

   (E) It is a multiple of 12.

6. Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a \$90 rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves \$15 by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

   (A) 750 **(B)** 900 **(C)** 1000 **(D)** 1050 **(E)** 1500

7. While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the

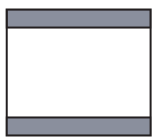
boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 10

8. What is the volume of a cube whose surface area is twice that of a cube with volume 1?

(A) $\sqrt{2}$ **(B)** 2 **(C)** $2\sqrt{2}$ **(D)** 4 **(E)** 8

9. Older television screens have an aspect ratio of 4:3. That is, the ratio of the width to the height is 4:3. The aspect ratio of many movies is not 4:3, so they are sometimes shown on a television screen by "letterboxing" — darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2:1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



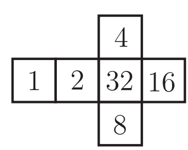
(A) 2 **(B)** 2.25 **(C)** 2.5 **(D)** 2.7 **(E)** 3

10. Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

(A) $\left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1$ **(B)** $\left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$ **(C)** $\left(\frac{1}{5}+\frac{1}{7}\right)t=1$

(D) $\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$ **(E)** $(5+7)t=1$

11. Three cubes are each formed from the pattern shown. They are then stacked on a table one on top of another so that the 13 visible numbers have the greatest possible sum. What is that sum?



(A) 154 **(B)** 159 **(C)** 164 **(D)** 167 **(E)** 189

12. A function f has domain $[0,2]$ and range $[0,1]$. (The notation $[a,b]$ denotes $\{x : a \le x \le b\}$.) What are the domain and range, respectively, of the function g defined by $g(x)=1-f(x+1)$?

(A) $[-1,1],[-1,0]$ **(B)** $[-1,1],[0,1]$ **(C)** $[0,2],[-1,0]$ **(D)** $[1,3],[-1,0]$
(E) $[1,3],[0,1]$

13. Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

14. What is the area of the region defined by the inequality $|3x-18|+|2y+7| \le 3$?

(A) 3 **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 5

15. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

16. The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log(b^n)$. What is n?

(A) 40 **(B)** 56 **(C)** 76 **(D)** 112 **(E)** 143

17. Let $a_1, a_2, \ldots$ be a sequence of integers determined by the rule $a_n = a_{n-1}/2$ if a_{n-1} is even and $a_n = 3a_{n-1}+1$ if a_{n-1} is odd. For how many positive integers $a_1 \le 2008$ is it true that a_1 is less than each of a_2, a_3, and a_4?

(A) 250 **(B)** 251 **(C)** 501 **(D)** 502 **(E)** 1004

18. Triangle ABC, with sides of length 5, 6, and 7, has one vertex on the positive x-axis, one on the positive y-axis, and one on the positive z-axis. Let O be the origin. What is the volume of tetrahedron $OABC$?

(A) $\sqrt{85}$ **(B)** $\sqrt{90}$ **(C)** $\sqrt{95}$ **(D)** 10 **(E)** $\sqrt{105}$

19. In the expansion of

$$\left(1+x+x^2+\cdots+x^{27}\right)\left(1+x+x^2+\cdots+x^{14}\right)^2,$$

what is the coefficient of x^{28}?

(A) 195 **(B)** 196 **(C)** 224 **(D)** 378 **(E)** 405

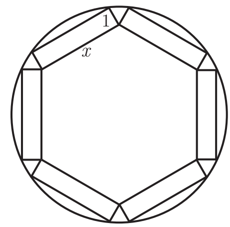
20. Triangle ABC has $AC = 3$, $BC = 4$, and $AB = 5$. Point D is on $\overline{AB}$, and $\overline{CD}$ bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

(A) $\frac{1}{28}(10-\sqrt{2})$ **(B)** $\frac{3}{56}(10-\sqrt{2})$ **(C)** $\frac{1}{14}(10-\sqrt{2})$ **(D)** $\frac{5}{56}(10-\sqrt{2})$
(E) $\frac{3}{28}(10-\sqrt{2})$

21. A permutation $(a_1, a_2, a_3, a_4, a_5)$ of (1, 2, 3, 4, 5) is heavy-tailed if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

(A) 36 **(B)** 40 **(C)** 44 **(D)** 48 **(E)** 52

22. A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?



(A) $2\sqrt{5}-\sqrt{3}$ **(B)** 3 **(C)** $\dfrac{3\sqrt{7}-\sqrt{3}}{2}$ **(D)** $2\sqrt{3}$
(E) $\dfrac{5+2\sqrt{3}}{2}$

23. The solutions of the equation $z^4+4z^3i-6z^2-4zi-i=0$ are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

(A) $2^{\frac{5}{8}}$ **(B)** $2^{\frac{3}{4}}$ **(C)** 2 **(D)** $2^{\frac{5}{4}}$ **(E)** $2^{\frac{3}{2}}$

24. Triangle ABC has $\angle C = 60^\circ$ and $BC = 4$. Point D is the midpoint of $\overline{BC}$. What is the largest possible value of $\tan(\angle BAD)$?

(A) $\dfrac{\sqrt{3}}{6}$ **(B)** $\dfrac{\sqrt{3}}{3}$ **(C)** $\dfrac{\sqrt{3}}{2\sqrt{2}}$ **(D)** $\dfrac{\sqrt{3}}{4\sqrt{2}-3}$ **(E)** 1

25. A sequence (a_1,b_1), (a_2,b_2), (a_3,b_3), $\ldots$ of points in the coordinate plane satisfies

$$(a_{n+1},b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n) \quad \text{for} \quad n = 1,2,3,\ldots.$$

Suppose that $(a_{100},b_{100}) = (2,4)$. What is a_1+b_1?

(A) $-\dfrac{1}{2^{97}}$ **(B)** $-\dfrac{1}{2^{99}}$ **(C)** 0 **(D)** $\dfrac{1}{2^{98}}$ **(E)** $\dfrac{1}{2^{96}}$

2008

AMC 12 – CONTEST A

DO NOT OPEN UNTIL

TUESDAY, February 12, 2008

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 12, 2008.** Nothing is needed from inside this package until February 12, 2008.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination during this period via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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2008 AIME

The 26th annual AIME will be held on Tuesday, March 18, 2008, with the alternate on Wednesday, April 2, 2008. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 29 and 30, 2008. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: www.unl.edu/amc.